PARAMETRIC TRICKS OF THE TRADE

1. **CHEESY PARAMETERIZATIONS OF GRAPHS OF FUNCTIONS:**

   - If $y = f(x)$ for $a \le x \le b$, let $x = t$ and $y = f(t)$ for $a \le t \le b$

     vector form: $\vec{r}(t) = \langle t, f(t) \rangle$, $a \le t \le b$.

   - If $x = g(y)$ for $c \le y \le d$, let $y = t$ and $x = g(t)$ for $c \le t \le d$

     vector form: $\vec{r}(t) = \langle g(t), t \rangle$, $c \le t \le d$.

2. **LINE SEGMENTS:** To parametrize the line segment *from* $P(x_0, y_0, z_0)$ *to* $Q(x_1, y_1, y_1)$:

   Let $x = x_0 + t\Delta x$, $y = y_0 + t\Delta y$, and $z = z_0 + t\Delta z$ for $0 \le t \le 1$.

   Here, $\Delta x = x_1 - x_0$, $\Delta y = y_1 - y_0$, and $\Delta z = z_1 - z_0$.

   vector form: $\vec{r}(t) = \langle x_0 + t\Delta x, y_0 + t\Delta y, z_0 + t\Delta z \rangle$, for $0 \le t \le 1$.

   To parametrize the entire line containing P and Q, drop the restriction on t: $-\infty < t < \infty$.

   NOTE 1: If $\vec{v}$ is parallel to the line and $\vec{r}_0 = \langle x_0, y_0, z_0 \rangle$ is a point on the line, then: $\vec{r}(t) = \vec{r}_0 + t\vec{v}$

   NOTE 2: For line / line segments in the plane, simply drop the z-component of the vector.

3. **CIRCLES AND ELLIPSES:** To parametrize $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, where $a, b > 0$, think:

   $$\cos^2(t) + \sin^2(t) = 1$$

   Let $\cos(t) = \dfrac{x-h}{a}$ and $\sin(t) = \dfrac{y-k}{b}$ which gives: $x = h + a\cos(t)$, $y = k + b\sin(t)$.

   vector form: $\vec{r}(t) = \langle h + a\cos(t), k + b\sin(t) \rangle$

   To trace out *once* around the circle/ellipse *counter-clockwise*, restrict t: $0 \le t < 2\pi$.

4. **HYPERBOLAS:** To parametrize $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$, where $a, b > 0$. think:

   $$\cosh^2(t) - \sinh^2(t) = 1$$

   Let $\cosh(t) = \dfrac{x-h}{a}$ and $\sinh(t) = \dfrac{y-k}{b}$ which gives: $x = h + a\cosh(t)$, $y = k + b\sinh(t)$.

   vector form: $\vec{r}(t) = \langle h + a\cosh(t), k + b\sinh(t) \rangle$.

   The range $-\infty < t < \infty$, traces out the *right branch* of the hyperbola.

Follow-up questions:

- How would you trace out the left branch of $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$?

- How would you parametrize $\dfrac{(y-k)^2}{b^2} - \dfrac{(x-h)^2}{a^2} = 1$?

- How could you exploit the identity: $\sec^2(t) - \tan^2(t) = 1$ instead of using hyperbolic functions?

ADJUSTING PARAMETERS

1. **REVERSING ORIENTATION:** Given $\vec{r}(t)$, $a \le t \le b$, define $\vec{r}_{\text{opp}}(t) = \vec{r}(-t)$, $-b \le t \le -a$.

   $\vec{r}_{\text{opp}}$ traces out the same curve as $\vec{r}$, just with the opposite orientation.

2. **SHIFTING:** Given $\vec{r}(t)$, $a \le t \le b$, define $\vec{r}_{\text{shift}}(t) = \vec{r}(t-c)$, $a+c \le t \le b+c$.

   $\vec{r}_{\text{shift}}$ traces out the same curve as $\vec{r}$, just with the parameter shifted c units ahead.

PRACTICE PROBLEMS:
For each of the following paths below, find a vector valued function that traces out the given curve with the implied orientation. Unless otherwise directed, shift your parameter t so that t begins at 0.

1. Let C be the circle of radius 1 centered at $(1,0)$. Find a parametrization of the circumference of C starting at the point $(2,0)$ and proceeding counter-clockwise.

   Ans: $\vec{r}(t) = \langle 1+\cos(t), \sin(t)\rangle$, $0 \le t < 2\pi$.

2. Find a parametrization of the two part path starting at $(0,1)$ following $y = x^2+1$ to $(1,2)$ and then proceeding along a line segment to $(3,0)$. Be sure to shift the parameter so the second path starts when the first path stops.

   $$\text{Ans: } \vec{r}(t) = \begin{cases} \langle t, t^2+1\rangle, & \text{if } 0 \le t \le 1 \\ \langle -1+2t, 4-2t\rangle, & \text{if } 1 \le t \le 2 \end{cases}$$

3. Find a parametrization of the oriented triangle in the first octant which begins at $(1,0,0)$, proceeds to $(0,1,0)$, then to $(0,0,1)$ and, finally, back to $(1,0,0)$.

   $$\text{Ans: } \vec{r}(t) = \begin{cases} \langle 1-t, t, 0\rangle, & \text{if } 0 \le t \le 1 \\ \langle 0, 2-t, t-1\rangle, & \text{if } 1 \le t \le 2 \\ \langle t-2, 0, 3-t\rangle, & \text{if } 2 \le t \le 3 \end{cases}$$